

Proof of Correspondence between Keys and Encoding Maps in an Authentication Code

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Abstract. In a former paper the authors introduced two new systematic authentication codes based on the Gray map over a Galois ring. In this paper, it is proved the one-to-one onto correspondence between keys and encoding maps for the second introduced authentication code.

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1 Introduction

Systematic authentication codes without secrecy were defined in [1]. In [2] two new systematic authentication codes based on the Gray map on a Galois ring are introduced with the purpose of optimally reducing the impersonation and substitution probabilities. The first code is another example of a previously constructed code using the Gray map on Galois rings and modules over these rings [3,4]. The second code generalises the construction in [3], on the assumption of the existence of an appropriate class of bent functions. For this code, the existence of the bijection between the key space and the set of encoding maps is proved in this paper in a rather long but exhaustive way.

2 Refreshment of basic notions

2.1 General systematic authentication codes

We recall that a *systematic authentication code without secrecy* [1] is a structure (S, T, K, E) where S is the *source state space*, T is the *tag space*, K is the *key space* and $E = (e_k)_{k \in K}$ is a sequence of *encoding rules* $S \rightarrow T$.

A *transmitter* and a *receiver* agree to a secret key $k \in K$. Whenever a source $s \in S$ must be sent, the participants proceed according to the following protocol:

Transmitter	Receiver
evaluates $t = e_k(s) \in T$	
forms the pair $m = (s, t)$	\xrightarrow{m} receives $m' = (s', t')$, evaluates $t'' = e_k(s') \in T$ if $t' = t''$ then accepts s' , other- wise the message m' is rejected

The communicating channel is public, thus it can be eavesdropped upon by an *intruder* able to perform either *impersonation* or *substitution* attacks through the public channel. The intruder's success probabilities for impersonation and substitution are, respectively [5]

$$p_I = \max_{(s,t) \in S \times T} \frac{|\{k \in K \mid e_k(s) = t\}|}{|K|} \quad (1)$$

$$p_S = \max_{(s,t) \in S \times T} \max_{(s',t') \in (S - \{s\}) \times T} \frac{|\{k \in K \mid e_k(s) = t \text{ \& } e_k(s') = t'\}|}{|\{k \in K \mid e_k(s) = t\}|} \quad (2)$$

2.2 The second systematic authentication code

The second systematic authentication code introduced in [2] is constructed as follows:

Let p be a prime number, $r, \ell, n \in \mathbb{Z}^+$ and $q = p^\ell$. Let $A = \text{GR}(p^r, \ell)$ and $B = \text{GR}(p^r, \ell n)$ be the corresponding Galois rings of degrees ℓ and ℓn . We denote by $T(A) = \{0\} \cup \left(\xi_A^j\right)_{j=0}^{q-2}$ the set of Teichmüller representatives of \mathbb{F}_q in A . Then $p^{r-1}A = \{ap^{r-1} \mid a \in T(A)\}$. We define $\Xi = (0, \rho(\xi_A), \dots, \rho(\xi_A^{q-2}), \rho(\xi_A^{q-1})) \in \mathbb{F}_q^q$ and $L = \{r_0 + r_1p + \dots + r_{r-2}p^{r-2} \mid r_0, \dots, r_{r-2} \in T(A)\} \subset A \setminus p^{r-1}A \cup \{0\}$. Since $\langle p^{r-1} \rangle = \{ap^{r-1} \mid a \in T(A)\}$, if $a, b \in L$ then $a - b \in A \setminus p^{r-1}A$.

Similarly, $T(B)$ is the set of the Teichmüller representatives of F_{q^m} in B .

Let $n \in \mathbb{Z}^+$ and $t \leq n$. For any $i < n$, we denote $e_i = (\delta_{ij})_{j=0}^{n-1}$ as the i -th “canonical” vector. For any $b \in T(B)^n$, let

$$X_{b,t} = \left\{ \sum_{j=0}^{t-2} b_j e_j, b_{t-1} e_{t-1}, \dots, b_{n-1} e_{n-1} \right\} \subset B^n, \quad (3)$$

$$N = \bigcup_{b \in T(B)^n} X_{b,t},$$

$$L = \left\{ \sum_{i=0}^{r-2} r_i p^i \mid (r_0, \dots, r_{r-2}) \in T(A)^{r-1} \right\}. \quad (4)$$

Then $|X_{b,t}| = n - t + 1$, $|N| = q^{m(t-1)} + (n - (t - 1))q^m$, $|L| = q^{r-1}$, $L \subset (A - p^{r-1}A) \cup \{0\}$ and also $\forall u, v \in L : (u - v) \in (A - p^{r-1}A) \cup \{0\}$. Let us consider an $(r - 1)n$ -subset of $T(A) - \{0, 1\}$,

$$\eta = \{\eta_k\}_{k=0}^{(r-1)n-1}, \quad (5)$$

and

$$D_\eta = \{(\eta_{(i-1)n+j}, p^i e_j) \mid 1 \leq i \leq r - 1, 0 \leq j \leq n - 1\}. \quad (6)$$

Then $D_\eta \subset A \times B^n$ and $|D_\eta| = (r - 1)n$.

Let us write $T(B) = \{0\} \cup (\xi_B^k)_{k=0}^{q^m-2}$, $G(T(B)) = \{\xi_B^k \mid \gcd(k, q^m - 1) = 1\}$ and $\theta = \{\theta_j\}_{j=0}^{n-1}$, which is an n -sequence of $G(T(B))$ (repetitions are allowed),

and $\zeta \in T(B) - \{0\}$. For each integer k , with $0 \leq k \leq q^m - (r-1)n - 2$, let

$$T_{\theta\zeta k} = \left\{ (\theta_j^i, (\zeta + \theta_j^i p^{1+(k \bmod (r-1))})e_j) \mid 0 \leq i \leq q^m - 2, 0 \leq j \leq n-1 \right\}.$$

Then $T_{\theta\zeta k} \subset B \times B^n$ and $|T_{\theta\zeta k}| = (q^m - 1)n$. Now, let $Z = \{\zeta_k\}_{k=0}^{q^m - (r-1)n - 2}$ be a subset of $T(B) - \{0\}$, with $(q^m - 1 - (r-1)n - 1)$ elements, such that $Z \cap \eta = \emptyset$, and

$$\mathbf{T}_{\eta\theta Z} = D_\eta \cup \bigcup_{k=0}^{q^m - (r-1)n - 2} T_{\theta\zeta k}. \quad (7)$$

Then $\mathbf{T}_{\eta\theta Z} \subset B \times B^n$ and

$$\begin{aligned} |\mathbf{T}_{\eta\theta Z}| &= (r-1)n + (q^m - 1 - (r-1)n)(q^m - 1)n \\ &= [(r-1) + [(q^m - 1) - (r-1)n](q^m - 1)]n \end{aligned}$$

Let f be a bent function on B such that uf is a bent function for any unit $u \in S$ and let Φ be the Gray map [2] on A . The proposed Systematic Authentication Code, $\mathcal{A} = (S, T, K, E)$, is the following:

$$\begin{aligned} S &:= (T(B) \times B - \{(0, 0)\}) \times L, \\ T &:= \mathbb{F}_q, \\ K &:= \mathbb{Z}_{q^{t(n+1)}}, \\ E &:= \{E_k(s) = pr_k(u_s), k \in K, s \in B\}. \end{aligned}$$

where for $s = (a, b, c) \in S$, $\beta \in p^{r-1}A = \{\beta_1, \beta_2, \dots, \beta_q\}$,

$$\begin{aligned} v_{s,\beta}(x) &= \beta + \text{Tr}_{B/A}(af(x) + bx) + c, \\ u_{s,\beta} &= (\Phi(v_{s,\beta}(x)))_{x \in B}, \\ u_s &= (u_{s,\beta})_{\beta \in p^{r-1}A}, \end{aligned}$$

and pr_k is the k -th projection map from $\mathbb{F}_q^{t(n+1)}$ onto \mathbb{F}_q , mapping u_s to its k -th coordinate.

For each $s = (s_0, s_1, s_2) \in S$ and each $w \in p^{r-1}A$, consider the map

$$\begin{aligned} v_{s,w} : B^n &\rightarrow A \\ x &\mapsto v_{s,w}(x) = \text{Tr}_{B/A}(s_0 f(x) + s_1 \cdot x) + s_2 + w \\ &= \gamma_{s_0 s_1} f(x) + s_2 + w \end{aligned} \quad (8)$$

Let

$$\begin{aligned} u_{s,w} &= (\Phi(v_{s,w}(x)))_{x \in B^n} \in \left(\mathbb{F}_q^{q^{r-1}}\right)^{q^{rmn}}, \\ u_s &= (u_{s,w})_{w \in p^{r-1}A} \in \left(\mathbb{F}_q^{q^{r-1}}\right)^{q^{rmn+1}}. \end{aligned} \quad (9)$$

Since $|p^{r-1}A| = q$, we have $\left(\mathbb{F}_q^{q^{r-1}}\right)^{q^{rmn+1}} \simeq \mathbb{F}_q^{q^{r(mn+1)}}$, thus we may assume $u_s \in \mathbb{F}_q^{q^{r(mn+1)}}$.

This paper is devoted to prove the following:

Theorem 1. *The map $K \rightarrow E$, $k \mapsto e_k$, is one-to-one.*

Proof. The theorem is clearly equivalent to the following statement:

$$\forall k_0, k_1 \in K : [k_0 \neq k_1 \implies \exists s \in S : \pi_{k_0}(u_s) \neq \pi_{k_1}(u_s)] \quad (10)$$

where u_s is given by relation (9), and $\pi_k(u_s)$ is the k -th entry of the element u_s .

According to (9), each element u_s , $s \in S$, is the concatenation of q arrays $u_{s,w}$, each of length $q^{r_{mn}}$. The index range $\{0, \dots, q^{r(mn+1)} - 1\}$ of the element u_s can be split as the concatenation of $q^{r_{mn+1}}$ integer intervals

$$K_{x,w} = \{\text{indexes of entries with the value } \Phi(v_{s,w}(x))\}$$

with $(x, w) \in B^n \times p^{r-1}A$, and each integer interval $K_{x,w}$ has length q^{r-1} .

We recall at this point that $|B^n \times p^{r-1}A| = q^{r_{mn}}q = q^{r_{mn+1}}$. Let $\alpha_b : B^n \rightarrow \{0, \dots, q^{r_{mn}} - 1\}$, $\alpha_a : p^{r-1}A \rightarrow \{0, \dots, q - 1\}$ be the corresponding natural bijections. Then we may identify

$$K_{x,w} \approx \{k \in K \mid k_{x,w}q^{r-1} \leq k \leq k_{x,w}q^{r-1} + (q^{r-1} - 1)\},$$

where

$$\forall (x, w) \in B^n \times p^{r-1}A : k_{x,w} = \alpha_b(x)q + \alpha_a(w). \quad (11)$$

Let $k_0, k_1 \in K \approx \{0, \dots, q^{r(mn+1)} - 1\}$ be two keys such that $k_0 \neq k_1$. Depending on the intervals $K_{x,w}$ in which these keys fall, we may consider four mutually disjoint and exhaustive cases.

- *Case I:* $\exists w \in p^{r-1}A, \exists x \in B^n : k_0 \in K_{x,w} \text{ \& } k_1 \in K_{x,w}$.
- *Case II:* $\exists w \in p^{r-1}A, \exists x, y \in B^n : x \neq y \text{ \& } k_0 \in K_{x,w} \text{ \& } k_1 \in K_{y,w}$.
- *Case III:* $\exists w_0, w_1 \in p^{r-1}A, \exists x \in B^n : w_0 \neq w_1 \text{ \& } k_0 \in K_{x,w_0} \text{ \& } k_1 \in K_{x,w_1}$.
- *Case IV:* $\exists w_0, w_1 \in p^{r-1}A, \exists x, y \in B^n :$

$$w_0 \neq w_1 \text{ \& } x \neq y \text{ \& } k_0 \in K_{x,w_0} \text{ \& } k_1 \in K_{y,w_1}.$$

The analysis of these cases, giving a full proof of the proposition, is rather extensive and it is provided in the following section.

3 Proof of Proposition 1

The detailed proof of Proposition 1 is presented in this section. The plan of the proof is sketched as Plan 1. In what follows, we will list extensively all the assertions claimed in the proof plans.

Assertion 1 *Based on the condition underlying statement I in Plan 1, the claim (10) holds.*

```

if Case I holds then
  | I. See Assertion 1
else
  | if Case II holds then
    | let  $k_{00} = k_0 - k_{x,w}$  and  $k_{10} = k_1 - k_{y,w}$  ;
    | if  $k_{00} = k_{10}$  then
      | | proceed as in Plan 2
    | else
      | | proceed as in Plan 3
    | end
  | else
    | if Case III holds then
      | let  $k_{00} = k_0 - k_{x,w_0}$  and  $k_{10} = k_1 - k_{x,w_1}$ , according to (11) ;
      | if  $k_{00} = k_{10}$  then
        | | III.0 See Assertion 11
      | else
        | | pick  $(s_0, s_1) \in \{0\} \times (N - \{0\})$  arbitrarily ;
        | | if  $\pi_{k_{00}} \circ \Phi(\text{Tr}_{B/A}(s_0 f(x) + s_1 \cdot x) + w_0) =$ 
        | |  $\pi_{k_{10}} \circ \Phi(\text{Tr}_{B/A}(s_0 f(x) + s_1 \cdot x) + w_1)$  then
        | | | III.1.0 See Assertion 12
        | | else
        | | | III.1.1 See Assertion 13
        | | end
      | end
    | else
      | (at this point, Case IV necessarily does hold )
      | let  $k_{00} = k_0 - k_{x,w_0}$  and  $k_{10} = k_1 - k_{x,w_1}$ , according to (11) ;
      | if  $\pi_{k_{00}} \circ \Phi(\text{Tr}_{B/A}(f(x))) = \pi_{k_{10}} \circ \Phi(\text{Tr}_{B/A}(f(y)))$  then
        | | IV.0 See Assertion 14
      | else
        | | IV.1 See Assertion 15
      | end
    | end
  | end
end

```

Plan 1. Plan of the proof of Proposition 1.

choose $j \in \{0, \dots, n-1\}$ such that the j -th entry of $x - y$ is not zero,
namely $x_j - y_j \neq 0$;
if $x_j - y_j \in p^{r-1}B - \{0\}$ **then**
| **II.0.0** See Assertion 2
else
| there are $\theta \in T_B - T_A$, and $t \leq r-1$ such that
| $\text{Tr}_{B/A}(\theta p^t(x_j - y_j)) \in p^{r-1}A - \{0\}$;
| **if** $\text{Tr}_{B/A}(x_j) = \text{Tr}_{B/A}(y_j)$ **then**
| | let $\zeta \in T_A - \{0\}$ be such that $(\theta, (\zeta + \theta p^t)e_j) \in \bigcup_{k=0}^{q^m - (r-1)n-2} T_{\theta \zeta_k k}$
| | as defined at (7) ;
| | we have
| |
$$\text{Tr}_{B/A}(\zeta x_j) = \sum_{k=0}^{r-1} d_k p^k = \text{Tr}_{B/A}(\zeta y_j)$$

| |
$$\text{Tr}_{B/A}(\theta p^t x_j) = \sum_{k=0}^{r-2} a_k p^k + a_{r-1} p^{r-1}$$

| |
$$\text{Tr}_{B/A}(\theta p^t y_j) = \sum_{k=0}^{r-2} a_k p^k + b_{r-1} p^{r-1}$$

| | with $a_{r-1} \neq b_{r-1}$;
| | let $(s_0, s_1) = (\theta, (\zeta + \theta p^t)e_j)$;
| | **if** $\pi_{k_{00}} \circ \Phi(\text{Tr}_{B/A}(\theta f(x))) = \pi_{k_{10}} \circ \Phi(\text{Tr}_{B/A}(\theta f(y)))$ **then**
| | | **II.0.1.0.0** See Assertion 3
| | | **else**
| | | | **II.0.1.0.1** See Assertion 4
| | | **end**
| | **else**
| | | **II.0.1.1** **if** $\pi_{k_{00}} \circ \Phi(\text{Tr}_{B/A}(f(x))) = \pi_{k_{10}} \circ \Phi(\text{Tr}_{B/A}(f(y)))$ **then**
| | | | There is a t , $0 \leq t \leq r-1$, such that
| | | | $\text{Tr}_{B/A}(p^t(x_j - y_j)) \in p^{r-1}A - \{0\}$ (here the hypothesis
| | | | $\text{Tr}_{B/A}(x_j) \neq \text{Tr}_{B/A}(y_j)$ is very important) ;
| | | | **if** $t = 0$ **then**
| | | | | **II.0.1.1.0.0** See Assertion 5
| | | | | **else**
| | | | | | **II.0.1.1.0.1** See Assertion 6
| | | | | **end**
| | | | **else**
| | | | | **II.0.1.1.1** See Assertion 7
| | | | **end**
| | | **end**
| | **end**
| **end**
end

Plan 2. First branch of Case II.

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let  $j \in \{0, \dots, n-1\}$  be such that  $x_j - y_j \neq 0$  ;
if  $x_j - y_j \in p^{r-1}B - \{0\}$  then
  | II.1.0 See Assertion 8
else
  | there exist  $\theta \in (T_B - T_A) \cup \{1\}$  and  $t \in \{1, \dots, r-1\}$  such that
  |  $\text{Tr}_{B/A}(\theta p^t(x_j - y_j)) \in p^{r-1}A - \{0\}$  ;
  | if  $\text{Tr}_{B/A}(x_j) = \text{Tr}_{B/A}(y_j)$  then
  | | let  $\zeta \in T_A - \{0\}$  be such that the pair  $(s_0, s_1) = (\theta, (\zeta + \theta p^t)e_j)$  is
  | | included in the set  $\bigcup_{k=0}^{q^m - (r-1)n-2} T_{\theta \zeta_k k}$  as defined at (7) ;
  | | if  $\Phi(\text{Tr}_{B/A}(\theta f(x))) = \Phi(\text{Tr}_{B/A}(\theta f(y)))$  then
  | | | II.1.1.0.0 See Assertion 9
  | | else
  | | | II.1.1.0.1 See Assertion 10
  | | end
  | else
  | | proceed as in statement II.0.1.1 of Plan 2
  | end
end

```

Plan 3. Second branch of Case II.

Proof. Let $(s_0, s_1) \in \{0\} \times (N - \{0\})$ and

$$\text{Tr}_{B/A}(s_0 f(x) + s_1 \cdot x) = \sum_{i=0}^{r-2} a_i p^i + a_{r-1} p^{r-1}.$$

For each $k \in \{0, \dots, r-2\}$, there exists $y^{(k)} = \sum_{i=0}^{r-2} y_{ik} p^i \in L$ such that

$$\text{Tr}_{B/A}(s_0 f(x) + s_1 \cdot x) + y^{(k)} = \begin{cases} a_k p^k + a_{r-1} p^{r-1} & \text{if } a_k \neq 0 \\ y_{kk} p^k + a_{r-1} p^{r-1} & \text{if } a_k = 0 \text{ \& } y_{kk} \neq 0 \end{cases}$$

Thus,

$$\begin{aligned} & \Phi\left(\text{Tr}_{B/A}(s_0 f(x) + s_1 \cdot x) + y^{(k)} + w\right) \\ &= \begin{cases} \Phi(a_k p^k) + \Phi(a_{r-1} p^{r-1} + w) & \text{if } a_k \neq 0 \\ \Phi(y_{kk} p^k) + \Phi(a_{r-1} p^{r-1} + w) & \text{if } a_k = 0 \text{ \& } y_{kk} \neq 0 \end{cases} \end{aligned}$$

We have that $(s_0, s_1, y^{(k)}) \in S$ and $w \in p^{r-1}A$.

Now, let $k_{00} = k_0 - k_{x,w}$ and $k_{10} = k_1 - k_{x,w}$. Let us consider the following possibilities:

- $q \nmid (k_{10} - k_{00})$: By taking $a_{r-2} \neq 0$, all other coefficients zero, and $s = (s_0, s_1, s_2)$, the k_{00} -projection of $u_{s,w}$ (see (9)) differs from its k_{10} -projection, thus $\pi_{k_0}(u_s) \neq \pi_{k_1}(u_s)$.
- $q \mid (k_{10} - k_{00})$ and $(\exists d: 1 \leq d \leq r-1 \text{ \& } q^{d-1} \leq k_{10} - k_{00} < q^d)$: By taking $a_{r-2-d} \neq 0$ and all other coefficients zero, and $s = (s_0, s_1, s_2)$, the k_{00} -projection of $u_{s,w}$ differs from its k_{10} -projection, thus $\pi_{k_0}(u_s) \neq \pi_{k_1}(u_s)$.

Assertion 2 Based on the condition underlying statement **II.0.0** in Plan 2, the claim (10) holds.

Proof. There exists $\theta \in T_B$ such that $\text{Tr}_{B/A}(\theta(x_j - y_j)) \in p^{r-1}B - \{0\}$. We express in their p -adic forms $\text{Tr}_{B/A}(\theta x_j)$ and $\text{Tr}_{B/A}(\theta y_j)$, namely

$$\text{Tr}_{B/A}(\theta x_j) = \sum_{k=0}^{r-1} a_k p^k, \quad \text{Tr}_{B/A}(\theta y_j) = \sum_{k=0}^{r-1} b_k p^k. \quad (12)$$

Thus

$$\sum_{k=0}^{r-1} (a_k - b_k) p^k = (a_0 - b_0) + \sum_{k=1}^{r-1} (a_k - b_k) p^k \in p^{r-1}A - \{0\}$$

and $a_0 - b_0 = 0$. Also

$$\sum_{k=1}^{r-1} (a_k - b_k) p^{k-1} = (a_1 - b_1) + \sum_{k=2}^{r-1} (a_k - b_k) p^{k-1} \in p^{r-2}A - \{0\}$$

and $a_1 - b_1 = 0$. Successively, continuing with this procedure, $\forall k \leq r-2$, $a_k = b_k$, and $(a_{r-1} - b_{r-1})p \in pA - \{0\}$. Hence $a_{r-1} \neq b_{r-1}$, and $\Phi(\text{Tr}_{B/A}(\theta x_j)) \neq \Phi(\text{Tr}_{B/A}(\theta y_j))$.

Let $s_0 = 0$, $s_1 = \theta e_j$, $s_2 = 0$ and $s = (s_0, s_1, s_2) \in S$. Then, according to (8),

$$\begin{aligned} \Phi(v_{s,w}(x)) &= \Phi(\text{Tr}_{B/A}(s_0 f(x) + s_1 \cdot x) + s_2 + w) \\ &= \Phi(\text{Tr}_{B/A}(\theta x_j)) + \Phi(w) \\ &\neq \Phi(\text{Tr}_{B/A}(\theta y_j)) + \Phi(w) \\ &= \Phi(\text{Tr}_{B/A}(s_0 f(y) + s_1 \cdot y) + s_2 + w) \\ &= \Phi(v_{s,w}(y)), \end{aligned}$$

and, in particular, $\pi_{k_{00}} \circ \Phi(v_{s,w}(x)) \neq \pi_{k_{10}} \circ \Phi(v_{s,w}(x))$. Thus, implication (10) holds under these conditions.

Assertion 3 Based on the condition underlying statement **II.0.1.0.0** in Plan 2, implication (10) holds.

Proof. Let $s_2 = d_{r-1}p^{r-1} + a_{r-1}p^{r-1} - \text{Tr}_{B/A}((\zeta + \theta p^t)x_j) = d_{r-1}p^{r-1} + b_{r-1}p^{r-1} - \text{Tr}_{B/A}((\zeta + \theta p^t)y_j)$. Then,

$$\begin{aligned} \Phi(v_{s,w}(x)) &= \Phi(\text{Tr}_{B/A}(s_0 f(x) + s_1 \cdot x) + s_2 + w) \\ &= \Phi(\text{Tr}_{B/A}(\theta f(x) + (\zeta + \theta p^t)x_j) + s_2 + w) \\ &= \Phi(\text{Tr}_{B/A}(\theta f(x)) + d_{r-1}p^{r-1} + a_{r-1}p^{r-1} + w) \\ &= \Phi(\text{Tr}_{B/A}(\theta f(x))) + \Phi(d_{r-1}p^{r-1}) + \Phi(a_{r-1}p^{r-1}) + \Phi(w). \end{aligned}$$

Thus

$$\Phi(v_{s,w}(y)) = \Phi(\text{Tr}_{B/A}(\theta f(y))) + \Phi(d_{r-1}p^{r-1}) + \Phi(b_{r-1}p^{r-1}) + \Phi(w),$$

hence $\Phi(v_{s,w}(x)) \neq \Phi(v_{s,w}(y))$. In particular, $\pi_{k_{00}} \circ \Phi(v_{s,w}(x)) \neq \pi_{k_{10}} \circ \Phi(v_{s,w}(x))$. Thus, implication (10) holds under these conditions.

Assertion 4 Based on the condition underlying statement **II.0.1.0.1** in Plan 2, implication (10) holds.

Proof. Let $\theta \in T_B$ be as in Assertion 2 above and $(s_0, s_1, s_2) = (\theta, 0, 0)$. Then, $\Phi(v_{s,w}(x)) = \Phi(\text{Tr}_{B/A}(\theta f(x)) + \Phi(w)$ and $\Phi(v_{s,w}(y)) = \Phi(\text{Tr}_{B/A}(\theta f(y)) + \Phi(w)$. Hence $\pi_{k_{00}} \circ \Phi(v_{s,w}(x)) \neq \pi_{k_{10}} \circ \Phi(v_{s,w}(y))$.

Assertion 5 Based on the condition underlying statement **II.0.1.1.0.0** in Plan 2, implication (10) holds.

Proof. Let $s_0 = 0, s_1 = e_j, s_2 = 0$ and $s = (s_0, s_1, s_2) \in S$. Then as in Assertion 2 we conclude that $\pi_{k_{00}} \circ \Phi(v_{s,w}(x)) \neq \pi_{k_{10}} \circ \Phi(v_{s,w}(y))$.

Assertion 6 Based on the condition underlying statement **II.0.1.1.0.1** in Plan 2, implication (10) holds.

Proof. There is a pair $(s_0, s_1) = (\theta, p^t e_j)$ in the set D_η , as defined in (6), such that $\Phi(\text{Tr}_{B/A}(\theta f(x))) = \Phi(\text{Tr}_{B/A}(\theta f(y)))$, since $\theta \in T_A - \{0\}$. Written in p -adic form $\text{Tr}_{B/A}(p^t x_j) = \sum_{i=0}^{r-2} a_i p^i + a_{r-1} p^{r-1}$, $\text{Tr}_{B/A}(p^t y_j) = \sum_{i=0}^{r-2} a_i p^i + b_{r-1} p^{r-1}$ with $a_{r-1} \neq b_{r-1}$. An adequate selection of s_2 gives

$$\begin{aligned} \Phi(v_{s,w}(x)) &= \Phi(\text{Tr}_{B/A}(\theta f(x) + \text{Tr}_{B/A}(p^t x_j) + s_2 + w)) \\ &= \Phi(\text{Tr}_{B/A}(\theta f(x) + a_{r-1} p^{r-1} + w)) \\ &= \Phi(\text{Tr}_{B/A}(\theta f(x)) + \Phi(a_{r-1} p^{r-1}) + \Phi(w)). \end{aligned}$$

Similarly, $\Phi(v_{s,w}(y)) = \Phi(\text{Tr}_{B/A}(\theta f(y)) + \Phi(b_{r-1} p^{r-1}) + \Phi(w)$, and the right sides of the above identities are different, thus implication (10) holds in this case.

Assertion 7 Based on the condition underlying statement **II.0.1.1.1** in Plan 2, the claim (10) holds.

Proof. In this case, $\pi_{k_{00}} \circ \Phi(\text{Tr}_{B/A}(\eta_{(r-1)n} f(x))) \neq \pi_{k_{10}} \circ \Phi(\eta_{(r-1)n} \text{Tr}_{B/A}(f(y)))$ and there exists $\eta_{(r-1)n} \in T(A) - \{0\}$ such that $\eta_{(r-1)n}$ does not appear in η , because $(r-1)(n+1) < p^n - 1$. Now, we choose $s_1 = 0 \in B^n$, $s_2 = 0$ and $s = (\eta_{(r-1)n}, 0, 0)$. Then,

$$\begin{aligned} \pi_{k_{00}} \circ \Phi(v_{s,w}(x)) &= \pi_{k_{00}} \circ \Phi(\text{Tr}_{B/A}(s_0 f(x) + s_1 \cdot x) + s_2 + w) \\ &= \pi_{k_{00}} \circ \Phi(\text{Tr}_{B/A}(\eta_{(r-1)n} f(x)) + w) \\ &\neq \pi_{k_{10}} \circ \Phi(\text{Tr}_{B/A}(\eta_{(r-1)n} f(y)) + w) \\ &= \pi_{k_{10}} \circ \Phi(v_{s,w}(y)) \end{aligned}$$

and implication (10) holds.

Assertion 8 Based on the condition underlying statement **II.1.0** in Plan 3, implication (10) holds.

Proof. There is a $\theta \in T_B$ such that $\text{Tr}_{B/A}(\theta(x_j - y_j))j \in p^{r-1}A - \{0\}$. By writing $\text{Tr}_{B/A}(\theta x_j)$ and $\text{Tr}_{B/A}(\theta y_j)$ in p -adic form as in (12) we have that, as in Assertion 2, for any $i \leq r-2$, $a_i = b_i$ and $a_{r-1} - b_{r-1} \in p^{r-1}B - \{0\}$. Let $(s_0, s_1, s_2) = (0, \theta e_j, -\sum_{i=0}^{r-2} a_i p^i)$. Then $\Phi(v_{s,w}(x)) = \Phi(\text{Tr}_{B/A}(a_{r-1}p^{r-1}) + \Phi(w)$ and $\Phi(v_{s,w}(y)) = \Phi(\text{Tr}_{B/A}(b_{r-1}p^{r-1}) + \Phi(w)$. Hence $\pi_{k_{00}} \circ \Phi(v_{s,w}(x)) \neq \pi_{k_{10}} \circ \Phi(v_{s,w}(y))$.

Assertion 9 *Based on the condition underlying statement II.1.1.0.0 in Plan 3, implication (10) holds.*

Proof. There is a s_2 in $(T(B) - (\{0\} \cup \eta)) \times \{0\} \times L$ such that

$$\begin{aligned} \Phi(v_{s,w}(x)) &= \Phi(\text{Tr}_{B/A}(s_0 f(x) + s_1 \cdot x) + s_2 + w) \\ &= \Phi(\text{Tr}_{B/A}(\theta f(x) + (\zeta + \theta p^t)x_j) + s_2 + w) \\ &= \Phi(\text{Tr}_{B/A}(\theta f(x)) + \text{Tr}_{B/A}(\zeta x_j) + s_2 + \text{Tr}_{B/A}(\theta p^t x_j) + w) \\ &= \Phi(\text{Tr}_{B/A}(\theta f(x)) + c_{r-1}p^{r-1} + a_{r-1}p^{r-1} + w) \\ &= \Phi(\text{Tr}_{B/A}(\theta f(x))) + \Phi(c_{r-1}p^{r-1}) + \Phi(a_{r-1}p^{r-1}) + \Phi(w), \end{aligned}$$

where we have used the p -adic forms displayed in Plan 2.

Mutatis mutandis we get,

$$\Phi(v_{s,w}(y)) = \Phi(\text{Tr}_{B/A}(\theta f(y))) + \Phi(c_{r-1}p^{r-1}) + \Phi(b_{r-1}p^{r-1}) + \Phi(w),$$

hence $\Phi(v_{s,w}(x)) \neq \Phi(v_{s,w}(y))$. In particular, $\pi_{k_{00}} \circ \Phi(v_{s,w}(x)) \neq \pi_{k_{10}} \circ \Phi(v_{s,w}(x))$. Thus, implication (10) holds under these conditions.

Assertion 10 *Based on the condition underlying statement II.1.1.0.1 in Plan 3, implication (10) holds.*

Proof. We may proceed as in Assertion 4 to show that implication (10) holds under these conditions.

Assertion 11 *Based on the condition underlying statement III.0 in Plan 1, implication (10) holds.*

Proof. For any $s = (s_0, s_1, s_2) \in S$ we have

$$\begin{aligned} \Phi(v_{s,w_0}(x)) - \Phi(v_{s,w_1}(x)) &= \Phi(\text{Tr}_{B/A}(s_0 f(x) + s_1 \cdot x) + s_2 + w_0) \\ &\quad - \Phi(\text{Tr}_{B/A}(s_0 f(x) + s_1 \cdot x) + s_2 + w_1) \\ &= \Phi(w_0) - \Phi(w_1) \neq 0. \end{aligned}$$

In particular, $\pi_{k_{00}} \circ \Phi(v_{s,w_0}(x)) \neq \pi_{k_{00}} \circ \Phi(v_{s,w_1}(x))$. Thus, implication (10) holds in this case as well.

Assertion 12 *Based on the condition underlying statement III.1.0 in Plan 1, the claim (10) holds.*

Proof. If, written in its p -adic form, $\text{Tr}_{B/A}(s_0 f(x) + s_1 \cdot x) = \sum_{i=0}^{r-1} a_i p^i$, let $s_2 = -\sum_{i=0}^{r-2} a_i p^i$. As in Assertion 8, we will have

$$\pi_{k_{00}} \circ \Phi(v_{s,w_0}(x)) \neq \pi_{k_{10}} \circ \Phi(v_{s,w_1}(x)).$$

Assertion 13 *Based on the condition underlying statement III.1.1 in Plan 1, implication (10) holds.*

Proof. Let $s_2 = 0$. We will have $\pi_{k_{00}} \circ \Phi(v_{s,w_0}(x)) \neq \pi_{k_{10}} \circ \Phi(v_{s,w_1}(x))$.

Assertion 14 *Based on the condition underlying statement IV.0 in Plan 1, implication (10) holds.*

Proof. In this case, $\pi_{k_{00}} \circ \Phi(\text{Tr}_{B/A}(\eta_{(r-1)n} f(x))) = \pi_{k_{10}} \circ \Phi(\text{Tr}_{B/A}(\eta_{(r-1)n} f(y)))$ with $\eta_{(r-1)n} \in T(A) - \{0\}$ such that $\eta_{(r-1)n} \notin \eta$, where η is defined in (5).

If $(s_0, s_1, s_2) = (\eta_{(r-1)n}, 0, 0)$, then $\pi_{k_{00}} \circ \Phi(v_{s,w_0}(x)) \neq \pi_{k_{10}} \circ \Phi(v_{s,w_1}(x))$.

Assertion 15 *Based on the condition underlying statement IV.1 in Plan 1, implication (10) holds.*

Proof. Let $\eta \in T(A)$. Then, $\pi_{k_{00}} \circ \Phi(\eta \text{Tr}_{B/A}(f(x))) = \eta \pi_{k_{10}} \circ \Phi(\text{Tr}_{B/A}(f(x)))$ and $\pi_{k_{00}} \circ \Phi(\eta \text{Tr}_{B/A}(f(y))) = \eta \pi_{k_{10}} \circ \Phi(\text{Tr}_{B/A}(f(y)))$, and if there exists $\eta \in T(A)$ such that

$$\pi_{k_{00}} \circ \Phi(w_0) + \pi_{k_{00}} \circ \Phi(\eta \text{Tr}_{B/A}(f(x))) = \pi_{k_{10}} \circ \Phi(w_1) + \pi_{k_{10}} \circ \Phi(\eta \text{Tr}_{B/A}(f(y)))$$

then this element η is unique.

Let us choose $\zeta = \{\zeta_k\}_{k=0}^{q^m - (r-1)n-2}$, as was done in relation (7). Thus, either

$$\pi_{k_{00}} \circ \Phi(w_0) + \pi_{k_{00}} \circ \Phi(\zeta_k \text{Tr}_{B/A}(f(x))) \neq \pi_{k_{10}} \circ \Phi(w_1) + \pi_{k_{10}} \circ \Phi(\zeta_k \text{Tr}_{B/A}(f(y)))$$

or

$$\pi_{k_{00}} \circ \Phi(w_0) + \pi_{k_{00}} \circ \Phi(\zeta_{k'} \text{Tr}_{B/A}(f(x))) \neq \pi_{k_{10}} \circ \Phi(w_1) + \pi_{k_{10}} \circ \Phi(\zeta_{k'} \text{Tr}_{B/A}(f(y))),$$

where $\zeta_k, \zeta_{k'} \in T(A) \cap \zeta$, $k \neq k'$. Let j be an index witnessing the relations above and $(s_0, s_1, s_2) = (\eta_j, 0, 0)$. Then $\pi_{k_{00}} \circ \Phi(v_{s,w_0}(x)) \neq \pi_{k_{10}} \circ \Phi(v_{s,w_1}(x))$.

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